

# $SU(2)$ Ginzburg-Landau theory for degenerate Fermi gases with synthetic non-Abelian gauge fields

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The non-Abelian gauge fields play a key role in achieving novel quantum phenomena in condensed-matter and high-energy physics. Recently, the synthetic non-Abelian gauge fields have been created in the neutral degenerate Fermi gases, and moreover, generate many exotic effects. All the previous predictions can be well understood by the microscopic Bardeen-Cooper-Schrieffer theory. In this work, we establish an  $SU(2)$  Ginzburg-Landau theory for degenerate Fermi gases with the synthetic non-Abelian gauge fields. We firstly address a fundamental problem how the non-Abelian gauge fields, imposing originally on the Fermi atoms, affect the pairing field with no extra electric charge by a local gauge-field theory, and then obtain the first and second  $SU(2)$  Ginzburg-Landau equations. Based on these obtained  $SU(2)$  Ginzburg-Landau equations, we find that the superfluid critical temperature of the intra- (inter-) band pairing increases (decreases) linearly, when increasing the strength of the synthetic non-Abelian gauge fields. More importantly, we predict a novel  $SU(2)$  non-Abelian Josephson effect, which can be used to design a new atomic superconducting quantum interference device.

The non-Abelian gauge fields, whose different components do not commute each other, are a central building block of the theory of fundamental interactions. Attributed to their high degrees of controllability, tunability, and versatility, ultracold quantum gases are a powerful platform to simulate the non-Abelian gauge fields. In general, the atomic quantum gases are charge neutral, and are thus not influenced by external gauge fields the way electrons are. Fortunately, by controlling different laser-atom interactions, the synthetic non-Abelian gauge fields can be created in these neutral quantum gases [1–3]. Moreover, the simplest non-Abelian gauge field, which is always called the one-dimensional (1D) equal-Rashba-Dresselhaus(ERD)-type spin-orbit coupling, has been realized experimentally [4–12], using a pair of Raman lasers. Recently, the similar but spatial-dependent gauge field has also been achieved in ultracold  $^{87}\text{Rb}$  atom [13]. These important experiments pave a new way for exploring nontrivial quantum effects, induced by the synthetic non-Abelian gauge fields, in ultracold quantum gases. For instance, based on the microscopic Bardeen-Cooper-Schrieffer (BCS) theory, exotic superfluids [14–31], including the topological BCS [22–26] and Fulde-Ferrell-Larkin-Ovchinnikov phases [27–31], have been predicted in degenerate Fermi gases.

In the conventional charge superconductors, the  $U(1)$  Ginzburg-Landau (GL) theory, in parallel with the microscopic BCS theory, is another famous theory to explore relevant physics [32]. One of its most powerful features that it can be used to quantitatively describe the effects induced thermal fluctuations in the intermediate and strong coupling normal states, which are, however, missed in the BCS theory [33]. Moreover, some novel quantum phenomena, such as Josephson effect, flux flow,

and the melting of the Abrikosov vortex lattice, *etc.* [34], have also been revealed by this theory. However, the GL theory for degenerate Fermi gases with the synthetic non-Abelian gauge fields is still lacking. In this work, we establish an  $SU(2)$  GL theory for this system, based on the non-Abelian properties of the synthetic gauge fields.

Notice that in the conventional charge superconductors, the pairing has the electric charge  $2e$ , and is thus affected easily by the external gauge fields. However, the formed pairing in degenerate Fermi gases is charge neutral. It is natural to ask a fundamental and very important problem how the neutral pairing field interacts with the synthetic non-Abelian gauge fields, imposing originally on the Fermi atoms. We firstly address this key issue by a local gauge-field theory of the pairing field. Then, we obtain the first and second  $SU(2)$  GL equations by the variation of the total free energy with respect to the pairing field and the synthetic non-Abelian gauge fields. Based on these obtained  $SU(2)$  GL equations, we find that the superfluid critical temperature of the intra- (inter-) band pairing increases (decreases) linearly, when increasing the strength of the synthetic non-Abelian gauge fields. More importantly, we predict a novel  $SU(2)$  non-Abelian Josephson effect, which can be used to design a new atomic superconducting quantum interference device.

## Results

**Total free energy in space.** In general, the pairing field, resulting from the two-component Fermi atom field  $\phi(\mathbf{r})$  coupled with the synthetic  $SU(2)$  non-Abelian gauge fields, is expressed as  $\psi(\mathbf{R}) = \phi_1(\mathbf{r})\phi_2(\mathbf{r}')$ , where  $\phi_1$  and  $\phi_2$  are the fields for two different Fermi atoms,  $\mathbf{r}$  is the 3D space-dependent coordinate of the Fermi atom,

and  $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$  is the coordinate of the pairing field [35]. Obviously,  $\psi$  is a boson field. In terms of the local gauge-field theory of the pairing field (see Methods), we demonstrate strictly that this pairing field has an internal helical doublet and can interact with the same synthetic non-Abelian gauge fields, imposing originally on the Fermi atoms.

In addition, the total free energy is derived, in units of  $\hbar = c = 1$ , by (see Methods)

$$F_s = \int d^3\mathbf{R} (f_n + U_{\text{eff}} + f_c + f_G). \quad (1)$$

In equation (1),  $f_n$  is the energy density of the normal state.  $U_{\text{eff}} = a\psi^*\psi + b(\psi^*\psi)^2/2$ , with the coefficients  $a$  and  $b$ , is the effective potential of the pairing field. The first and second terms of  $U_{\text{eff}}$  are the free and self-interacting energy densities of the pairing field, respectively. The explicit expressions of the coefficients  $a$  and  $b$  can, in principle, be determined from the microscopic BCS theory [35]. In general, the coefficient  $a$  is dependent of temperature. When the temperature is lower than the superfluid critical temperature,  $a > 0$ , while  $a < 0$  vice versa. On contrary, the coefficient  $b$  is positive for any temperature.  $f_c = |\Pi_i\psi|^2/4m$  is the kinetic energy, where  $\Pi_i = -i\partial_i - \alpha A_i$  and  $m$  is the mass of the Fermi atom.  $f_G = \varrho L_{ij}L_{ij}$  is the energy density functional of the synthetic non-Abelian gauge fields  $A_i$ , where  $L_{ij} = \Pi_i A_j - \Pi_j A_i$  is the tensor of the synthetic non-Abelian gauge fields  $A_i$ , and satisfies the anti-symmetry property  $L_{ij} = -L_{ji}$ , and  $\varrho$  is a coefficient determined by the synthetic non-Abelian gauge fields  $A_i$ . In the previous discussions, the synthetic non-Abelian gauge fields are usually chosen, in the spin-basis representation, as

$$\mathbf{A} = [l\sigma_x, \chi l\sigma_y, 0, f(l)\hat{I}_2], \quad (2)$$

where  $l$  and  $f(l)$  are the introduced functions of space-time, the dimensionless constant  $\chi$  determines the type of the synthetic non-Abelian gauge fields, and  $\hat{I}_2$  is a  $2 \times 2$  unit matrix. For  $\chi = 1$ , the 2D RD-type non-Abelian gauge field emerges, and becomes the 1D ERD-type non-Abelian gauge field in the case of  $\chi = 0$  [4–12]. Recent experiment shows that the functions  $l$  and  $f(l)$  can be determined by the Rabi frequencies of laser fields [13], thus both space- and time- dependent functions  $l$  and  $f(l)$  can be accessible.

**The first  $SU(2)$  GL equation.** To describe the stable superfluid, we need study the variations of the total free energy,  $\delta F_s(\psi)$ ,  $\delta F_s(\psi^*)$ , and  $\delta F_s(A_i)$ . In the case of the three-component non-Abelian gauge fields  $A_i$ , the results are very complicate. For simplicity, here we only deal with the in-plane non-Abelian gauge fields, i.e.,  $A_z = 0$ . In such a case, we obtain the first  $SU(2)$  GL equation (see Methods)

$$\frac{1}{4m} [(-i\partial_\zeta - \alpha A_\zeta)^2 - \partial_z^2] \psi + a\psi + b\psi^2\psi = 0 \quad (3)$$

with  $\zeta = x, y$ .

The gauge-invariant field equation (3) fully describes the interplay between neutral superfluids and the synthetic non-Abelian gauge fields, when the temperature is lower than the superfluid critical temperature. It seems that this equation is similar to that of the  $U(1)$  case. In fact, the physics is quite different. Attributed to the  $SU(2)$  properties of the synthetic gauge fields, there are two kinds of superfluid states, including the positive and negative helical states. Moreover, they couple with each other and both of them are vectors in 2D Hilbert space of the helical basis. It means that equation (3) is a two-component coupled equation in 2D Hilbert space. In addition, in the  $U(1)$  case, the pairing is formed by two spin states. However, in the presence of the synthetic non-Abelian gauge fields, the pairing emerges in two helical states. These different spin and helical states lead to different dispersion relations, and thus different microscopic quantum statistics of the interacting many-body systems. It implies that the coefficients  $a$  and  $b$  are also different.

Due to existence of the term  $b\psi^2\psi$ , the two-component nonlinear equation (3) is hard to be solved exactly. Here we use an approximate linearization method (i.e., assuming  $b\psi^2\psi \simeq 0$ ) to deal with this equation [36, 37]. As an example, we consider a static RD-type non-Abelian gauge field, i.e.,  $l = 1$  and  $\chi = 1$  in equation (2). In such case, we rewrite the spatial part of this non-Abelian gauge field as  $[\sigma_x k_F(\xi_0 + \varkappa y), \sigma_z k_F(\xi_0 + \varkappa x), 0]$ , with the dimensionless infinitesimal  $\varkappa$  and the Fermi vector  $k_F$  of the non-interacting Fermi gases, and then assume the corresponding solution as  $\psi = \exp(ik_z z)h(x, y)$ . The introduced dimensionless infinitesimal  $\varkappa$  doesn't change the static property of the RD-type non-Abelian gauge field since  $\varkappa y \rightarrow 0$  and  $\varkappa x \rightarrow 0$ , but is an auxiliary quality, which only help us to approximately solve equation (3). Substituting the assumed solution  $\psi$  into equation (3) and using the approximate linearization method [36, 37], we obtain the following 2D oscillator-type equation:  $-(\partial_x^2 + \partial_y^2)h(x, y)/4m + m\varkappa^2[\omega_{cy}^2(y - C_0)^2 + \omega_{cx}^2(x - C_0)^2]h(x, y) = (|a| - k_z^2/4m)h(x, y)$ , where  $\omega_{cx} = \alpha\sigma_z k_F/2m$  and  $\omega_{cy} = \alpha\sigma_x k_F/2m$  are the circular frequencies in the  $x$  and  $y$  directions, respectively, and  $C_0 = -\xi_0/\varkappa$ . By further solving the above oscillator-type equation, we obtain  $|a| - k_z^2/4m = (n'_x + 1/2)\omega_{cx} + (n'_y + 1/2)\omega_{cy}$ , where  $n'_x$  and  $n'_y$  are the positive integers. When the condensate occurs, only the ground state ( $n'_x = n'_y = 0$ ,  $k_z = 0$ ) becomes significant [36, 37]. As a consequence, the critical temperature is obtained, in the spin-basis representation, by  $T_c^s \simeq T_c(0) - \alpha k_F(\sigma_x + \sigma_z)/4a_T m$ , where  $T_c(0)$  is the critical temperature without the synthetic non-Abelian gauge fields, and  $a_T(> 0)$  is the leading-order expansion coefficient of  $a$  at  $T_c(0)$ .

Since in this work we investigate the physics of superfluid with the helical doublet, the critical temperature is

obtained, from a transformation of  $SU(2)$  group representation to the helical basis of pairing doublet, by

$$T_c \simeq T_c(0) - \frac{\sqrt{2}\alpha k_F \sigma_z}{4a_T m}. \quad (4)$$

When  $\alpha = 0$ ,  $T_c = T_c(0)$ , as expected. Equation (4) shows that the superfluid critical temperature is a  $2 \times 2$  matrix, because equation (3) is a two-component coupled equation. The diagonal elements reflect the critical temperature for the different superfluid states (the positive and negative helical states). Using the similar consideration of the electric charge matrix of the left-handed doublet of lepton [38], we find that, when increasing the coupling strength  $\alpha$ , the critical temperature of the pairing field in the negative helical state increases linearly from a non-zero value, which is consistent with the result derived from the microscopic BCS theory with the Nozières–Schmitt–Rind correction [18]. Moreover, we can confirm that the pairing fields in the positive and negative helical states govern the superfluid physics of the inter- and intra- band pairings, respectively. For the superfluid critical temperature of the inter-band pairing, it decreases linearly when increasing the coupling strength  $\alpha$ . This behavior can also be easily understood since the inter-band pairing is gradually suppressed, attributed to the blocking effect in Fermi surface.

**The second  $SU(2)$  GL equation.** The second  $SU(2)$  GL equation is obtained by (see Methods)

$$\frac{i\alpha}{4m}(\psi^* \partial_\zeta \psi - \psi \partial_\zeta \psi^*) + \frac{\alpha^2}{2m} \psi^* \psi A_\zeta = 2\rho(\Lambda_\zeta - i2\alpha\Theta_\zeta), \quad (5)$$

where

$$\Lambda_\zeta = \partial_\eta \partial_\zeta A_\eta - \partial_\eta^2 A_\zeta - \partial_z^2 A_\zeta, \quad (6)$$

$$\Theta_\zeta = -(\partial_\zeta A_\eta) A_\eta + (\partial_\eta A_\zeta) A_\eta \quad (7)$$

with  $\eta = x, y$ . Equation (5) is also a two-component field equation. The left term of this equation reflects the in-plane supercurrents [35], i.e.,

$$j_\zeta = \frac{i\alpha}{4m}(\psi^* \partial_\zeta \psi - \psi \partial_\zeta \psi^*) + \frac{\alpha^2}{2m} \psi^* \psi A_\zeta, \quad (8)$$

This means that equation (5) governs the interplay between the in-plane supercurrents  $j_\zeta$  and the synthetic non-Abelian gauge fields  $A_\zeta$ . In addition, the term  $\Theta_\zeta$  in equation (7) is a new term, originating from the non-Abelian properties of the synthetic gauge fields  $A_\zeta$ . The supercurrent in the  $z$  direction is given by (see Methods)

$$j_z = \frac{i\alpha}{4m}(\psi^* \partial_z \psi - \psi \partial_z \psi^*). \quad (9)$$

In terms of Noether's theorem [39], the neutral supercurrents in equations (8) and (9) are the  $SU(2)$

charge currents, rather than the conventional probability currents ( $j_i = n_0 \partial_i \theta / 2m$ ) of superfluid order parameter  $\psi_0 = \sqrt{n_0} e^{i\theta}$  without any gauge field, where  $n_0$  is the density of pairing. However, the supercurrent in the  $z$  direction is trivial, since it doesn't interact with the synthetic non-Abelian gauge fields  $A_\zeta$ . When the synthetic gauge fields are the  $U(1)$  cases, the term  $\Theta_\zeta = 0$ , and equations (3) and (5) reduce respectively to  $[(-i\partial_\zeta - e' A_\zeta)^2 - \partial_z^2] \psi / 4m + a\psi + b\psi^2 \psi = 0$  and  $ie'(\psi^* \partial_\zeta \psi - \psi \partial_\zeta \psi^*) / 4m + e'^2 \psi^* \psi A_\zeta / 2m = 2\rho \Lambda_\zeta$ , where  $e'$  is the effective electric charge [35]. Moreover, the pairing field  $\psi$  is a single-component scalar field.

We emphasize that the nonlinear  $SU(2)$  GL equation (5) is gauge invariant, even if the terms  $\Lambda_\zeta$  and  $\Theta_\zeta$  are dependent of the synthetic non-Abelian gauge fields. Notice that for the static synthetic non-Abelian gauge fields,  $\Lambda_\zeta = 0$  and  $\Theta_\zeta = 0$ , derived from equations (6)-(7). It seems that the in-plane supercurrents vanish. In fact, in terms of  $SU(2)$  symmetry, we make a local gauge transformation  $\psi' = U_L \psi$  and  $A'_i = -i(\partial_i U_L) U_L^\dagger / \alpha + U_L A_i U_L^\dagger$ , and then obtain  $\Lambda'_\zeta \neq 0$  and  $\Theta'_\zeta \neq 0$ , i.e., equation (5) as well as the in-plane supercurrents still exist.

If the synthetic non-Abelian gauge fields are dependent of space-time, they can not be transformed to the static cases by a local gauge transformation. In this case, the superfluid physics becomes very rich, and however, is difficult to be discussed by the microscopic BCS theory. On contrary, our established GL theory is a powerful tool in this respect. In the Table I, we give the explicit expressions of  $\Lambda_\zeta$ ,  $\Theta_\zeta$ , and especially, the supercurrents for the 2D RD- and 1D ERD- type non-Abelian gauge fields with  $l = \Sigma t + \varepsilon \sin(\omega_0 t) / \omega_0$ , where the physical meanings of parameters  $\Sigma$ ,  $\varepsilon$ , and  $\omega_0$  will be interpreted in the following discussions. For the 1D ERD-type non-Abelian gauge field, the new term  $\Theta_\zeta$  disappears. In addition, we will show in the next section that the space-time-dependent non-Abelian gauge fields generate a novel  $SU(2)$  non-Abelian Josephson effect, which is a tunneling phenomenon in a weakly-linked superfluid system [40].

**$SU(2)$  non-Abelian Josephson effect.** To predict the  $SU(2)$  non-Abelian Josephson effect, we consider two identical degenerate Fermi gases without initial population imbalance, respectively distributed in two sides of double-well potential through a weakly-linked barrier [41–44], and denote these two regions as I and II (see Fig.1). The space-time-dependent non-Abelian gauge fields are chosen as the terms in equation (2). In this neutral Fermi atom system, we can investigate a gauge-invariant mass current  $j_\zeta^m = 2m j_\zeta$ , where the factor 2 originates that the formed pairing consists of two atoms [45, 46].

TABLE I: The explicit expressions of  $\Lambda_\zeta$ ,  $\Theta_\zeta$ , and especially, the supercurrents for the 2D RD- and 1D ERD- type non-Abelian gauge fields. The supercurrent 1 can be influenced by the synthetic non-Abelian gauge fields, while the supercurrent 2 is only a trivial  $SU(2)$  charge current, like Eq. (9). In the case of the 1D ERD-type non-Abelian gauge field, the new term  $\Theta_\zeta$  disappears, and the supercurrent 1 emerges only in the  $x$  direction. Here, the functions are defined as  $y_1 = \Sigma t^2 \partial_\eta \Sigma + t \partial_\eta (\Sigma \varepsilon) \sin(\omega_0 t) / \omega_0$  and  $y_2 = \varepsilon \partial_\eta \varepsilon \sin^2(\omega_0 t) / \omega_0^2$ , respectively. The indices  $\zeta$  and  $\eta$  satisfy the Einstein rule, in which  $\zeta$  and  $\eta$  only take different coordinates of  $x$  and  $y$  at the same time, i.e., if  $\zeta = x$ , then  $\eta = y$ .

	ERD	RD
$SU(2)$ gauge fields	$(l\sigma_x, 0, 0)$	$(l\sigma_x, l\sigma_y, 0)$
$\Lambda_\zeta$	$-(\partial_y^2 + \partial_z^2)l\sigma_x$	$-(\partial_\eta^2 + \partial_z^2)l\sigma_\zeta$
$\Theta_\zeta$	0	$(y_1 + y_2)(\sigma_\zeta \sigma_\eta - \hat{I})$
Supercurrent 1	$j_x^{\text{ERD}}$	$j_x^{\text{RD}}, j_y^{\text{RD}}$
Supercurrent 2	$j_y^{\text{ERD}}, j_z^{\text{ERD}}$	$j_z^{\text{RD}}$

When the vacuum condensate occurs, we have two conditions,  $\partial U_{\text{eff}} / \partial \psi = 0$  and  $\partial U_{\text{eff}} / \partial \psi^* = 0$  [39], and thus  $\psi^* \psi = -a/b$ . So the pairing field is written as  $\psi = (\sqrt{a/4b}, \sqrt{a/4b})^T e^{i(\frac{\varphi}{2} + \varphi)}$ , where  $\varphi$  is the phase. In addition, in this weakly-linked system, we also have two phenomenal boundary conditions,  $\partial \psi_I / \partial \zeta = \psi_{II} / d'$  and its complex conjugate, at the barrier [32], where the parameter  $d'$  is the width of barrier. Without the synthetic non-Abelian gauge fields  $A_\zeta$ , the direct-current Josephson mass current density in the  $\zeta$  direction is found as  $j_\zeta^m = -(a\alpha/2bd') \hat{\Xi} \sin(\Delta\varphi)$ , where the phase difference is defined as  $\Delta\varphi = \varphi_{II} - \varphi_I$  and  $\hat{\Xi}$  is a  $2 \times 2$  matrix with  $\hat{\Xi}_{11} = \hat{\Xi}_{12} = \hat{\Xi}_{21} = \hat{\Xi}_{22} = 1$ . In the presence of the synthetic non-Abelian gauge fields  $A_\zeta$ , the phase difference must be modified, in order to obtain the gauge-invariant mass current. By considering the dimensional property, we write a gauge-invariant phase difference as  $\Delta\varphi = \varphi_{II0} - \varphi_{I0} - \alpha \int_I^{II} A_\zeta d\zeta$ , where  $\Delta\Phi_0 = \varphi_{II0} - \varphi_{I0}$  is the initial phase difference between the regions I and II. Since the synthetic non-Abelian gauge fields  $A_\zeta$  are dependent of space-time, we further rewrite the total phase difference as  $\Delta\varphi = -\alpha \int_{t_0}^t dt \int_I^{II} \frac{dA_\zeta}{dt} d\zeta$ , and the mass current is thus given by

$$j_\zeta^m = -\frac{a\alpha}{2bd'} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sin(-\alpha \begin{bmatrix} 0 & \kappa_1 \\ \kappa_2 & 0 \end{bmatrix}), \quad (10)$$

where  $\kappa_1 = \int_{t_0}^t dt \int_I^{II} \lambda_1 \frac{d\zeta}{dt}$ ,  $\kappa_2 = \int_{t_0}^t dt \int_I^{II} \lambda_2 \frac{d\zeta}{dt}$ , and  $\lambda_1$  and  $\lambda_2$  are the dimensionless constants, determined by the type of the synthetic non-Abelian gauge fields. For the ERD-type ( $\chi = 0$ ) or the  $x$  component of the RD-type ( $\chi = 1$ ) non-Abelian gauge fields,  $\lambda_1 = \lambda_2 = 1$ ,

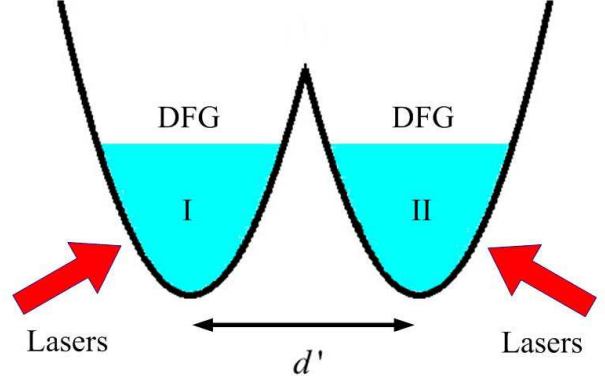


FIG. 1: A possible scheme to achieve the  $SU(2)$  non-Abelian Josephson effect and the Shapiro step. Two identical degenerate Fermi gases (DFGs) without initial population imbalance are respectively distributed in two sides of the double-well potential. The lasers are used to create the non-Abelian gauge fields.

which become  $\lambda_1 = -i$  and  $\lambda_2 = i$  in the case of the  $y$  component of the RD-type non-Abelian gauge field.

By controlling different laser-atom interactions (for example, adding a sinusoidal perturbation on Rabi frequencies, etc.) [1–3], we can choose  $dl/dt = \Sigma + \varepsilon \cos(\omega_0 t)$  with  $\Sigma = dG/d\zeta$  and  $\varepsilon = dg/d\zeta$ , where  $G$  and  $g$  reflect the chemical potential difference between the two wells of the unit  $SU(2)$  charge in the synthetic non-Abelian gauge fields and its amplitude of oscillating potential perturbation, respectively. In this case, the coefficient  $\varrho = \omega_0 d'^4/4$ . When  $\{G, g\} \ll E_F$  ( $E_F = k_F^2/2m$  is the Fermi energy of the non-interacting gases), equation (10) is simplified as (see Methods)

$$j_\zeta^m = -\frac{a\alpha}{2bd'} \hat{\Xi} \sum_{k=-\infty}^{\infty} \hat{\Omega}_k \quad (11)$$

with  $\hat{\Omega}_k = \text{diag}(\kappa_3, \kappa_4)$ , where  $\kappa_3 = \lambda_1 \lambda_2 (-1)^k J_k(\alpha g / \omega_0) \sin[(\alpha G - k\omega_0)t + \alpha \Delta\varphi_{01}]$ ,  $\kappa_4 = \lambda_1 \lambda_2 (-1)^k J_k(\alpha g / \omega_0) \sin[(\alpha G - k\omega_0)t + \alpha \Delta\varphi_{02}]$ ,  $J_k(\alpha g / \omega_0)$  is the  $k$ -th Bessel function with respect to  $\alpha g / \omega_0$ , and  $\Delta\varphi_{01}$  and  $\Delta\varphi_{02}$  are the matrix elements of the initial phase difference. Equation (11) shows that the space-time-dependent synthetic non-Abelian gauge fields can induce an alternating-current  $SU(2)$  Josephson mass current, which has never been predicted from the microscopic BCS theory [14–31].

In equation (11), if  $\alpha G - k\omega_0 = 0$ , the  $k$ -th current, with the magnitude  $I_k = |a\alpha J_k(\alpha g / \omega_0) / 2bd'|$ , converts to a direct current. This means that the Shapiro step, with the same magnitude  $I_k$ , emerges in our predicted mass-current Josephson effect. We define a  $k$ -th gap  $\Delta_k = \alpha G_0 - k\omega_0 = N\omega_0$ , where  $N \in \mathbb{Z}$  is a topological invariant of the fundamental group  $\pi_1(S^1)$  with



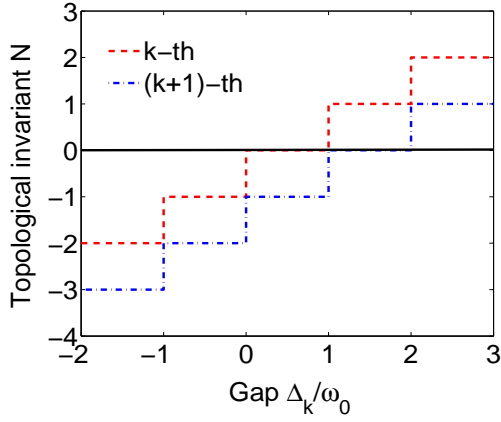


FIG. 2: The evolution of the topological invariant of the current component in equation (11). The black line represents the topologically-trivial mass current in the presence of a static Rashba-type non-Abelian gauge field.

$S^1 = \mathbb{R}^1 \cup \{\infty\}$ . We find that the direct-current component is topologically trivial ( $N = 0$ ) and the alternating-current component is topologically nontrivial ( $N \neq 0$ ). When increasing  $G$ , we need consider a generalized gap  $\Delta_k = (N + \sigma)\omega_0$ , where  $\sigma \in [0, 1]$ . When  $\sigma = 1$ , the  $k$ -th component becomes topologically nontrivial ( $N = 1$ ), while the  $(k + 1)$ -th component becomes topologically trivial ( $N = 0$ ), where a new Shapiro step, with the magnitude  $I_{k+1}$ , appears. This process is depicted in Fig. 2. We emphasize that our predictions arises from the space-time-dependent non-Abelian gauge fields. If the gauge field is chosen as the static Rashba-type gauge fields,  $\Delta_{\text{II,I}} \simeq d'$ , and only a constant Josephson mass current emerges. This Josephson mass current is topologically trivial, as shown in the black solid line in Fig. 2.

Finally, we briefly illustrate the possible experimental observation of the predicted  $SU(2)$  non-Abelian Josephson effect and the corresponding Shapiro steps. In experiments, the double-well potential can be constructed effectively by the superposition of a 1D periodic optical lattice with a 3D magnetic harmonic trap. The frequencies in the radial and normal directions of the 3D magnetic trap are of  $10^3\text{Hz}$  and  $10^2\text{Hz}$ , respectively [47]. The width and height of barrier are about  $2 \sim 4\mu m$  and of  $10^3\text{Hz}$ , respectively. When the pairing field condenses in the double-well potential with particle density  $n = 3 \times 10^{13}\text{cm}^{-3}$ ,  $G \sim 0.1E_F$  and  $g \sim 0.05E_F$  [48]. This means the condition  $\{G, g\} \ll E_F$  is valid. Thus, the predicted  $SU(2)$  non-Abelian Josephson effect as well as the Shapiro steps can be detected experimentally by the way of non-destructive phase contrast image [41].

## Discussion

In summary, we have demonstrated strictly that the

neutral pairing of degenerate Fermi gases interacts with the same synthetic non-Abelian gauge fields, imposing originally on the Fermi atoms. Moreover, we have obtained the first and second  $SU(2)$  GL equations, which allow us to predict new quantum effects, such as an  $SU(2)$  non-Abelian Josephson effect and the corresponding Shapiro steps for the space-time-dependent non-Abelian gauge fields. These results give new applications of the synthetic non-Abelian gauge fields. For example, we can design a novel atomic direct-current superconducting quantum interference device [49], based on the predicted  $SU(2)$  non-Abelian Josephson effect.

## Methods

**The local gauge theory of the pairing field.** In order to apply the local gauge theory, we in this subsection consider the 4D space-time-dependent coordinate, i.e.,  $x_\mu = (t, \mathbf{r})$ . We begin to study a two-component Fermi atom field  $\phi(x_\mu)$  coupled with the synthetic non-Abelian gauge field. When the massive Fermi atom field interacts with the synthetic non-Abelian gauge fields, its behavior is identical to a Dirac field with the same local gauge symmetry. In this Dirac-like atom field, each component reflects a spinor, corresponding to an internal helical state. The corresponding space-time action is written as [39]

$$S = \int d^4x_\mu [\bar{\phi} i \gamma^\mu (\partial_\mu + i\alpha A_\mu) \phi - m \bar{\phi} \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}]. \quad (12)$$

In equation (12),  $\gamma^\mu$  ( $\mu = 0, 1, 2, 3$ ) are the Dirac gamma matrices, satisfying the Clifford algebra  $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \delta_{\mu\nu} \hat{I}$ , where  $\delta_{\mu\nu}$  and  $\hat{I}$  are the Kronecker notation and  $4 \times 4$  unit matrix, respectively.  $\partial_\mu + i\alpha A_\mu$  are the covariant derivatives of the Fermi atom field  $\phi$ , where  $A_\mu(x_\mu)$  are the synthetic non-Abelian gauge fields with  $[A_\mu, A_\nu] \neq 0$ , and  $\alpha$  is a constant that governs the coupling between the Fermi atom field  $\phi$  and the non-Abelian gauge fields  $A_\mu$ .  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\alpha[A_\mu, A_\nu]$  is the tensor of our considered non-Abelian gauge fields. The space-time action in equation (12) is invariant via a local gauge transformation  $\phi' = U_L \phi$ , with  $U_L(x_\mu) = \exp[-i\Lambda^\epsilon(x_\mu)\tau_\epsilon]$ , where  $\tau_\epsilon$  ( $\epsilon = 1, 2, 3$ ) are the generators of the  $SU(2)$  Lie group, and  $\Lambda^\epsilon(x_\mu)$  are the phase factors of space-time.

For the pairing field, we firstly investigate the global gauge symmetry, and then generalize it to the local case. When we introduce a global  $SU(2)$  operator  $U_G = \exp(-i\Lambda_G^a \tau_a)$ , where  $\Lambda_G^a$  is independent of space-time, to make a gauge transformation  $\phi_1 \rightarrow U_G \phi_1$  or  $\phi_2 \rightarrow U_G \phi_2$ , the pairing field becomes  $\psi \rightarrow U_G \phi_1 \phi_2$ , which means that  $\psi' = U_G \psi$ . According to the principle of gauge-field theory, we should obtain a Lagrangian invariant  $\mathcal{L} = T - V$ , under the above global gauge transformation of the pairing field  $\psi$ . Using the relation  $-i\partial_\mu \psi' = -i\partial_\mu U_G \psi = -iU_G \partial_\mu \psi$  and its complex conjugate, we find directly that the kinetic energy  $T =$

$(i\partial^\mu\psi^*)(-i\partial_\mu\psi)$  is invariant. For the scalar pairing field  $\psi$  that can condense in a non-zero vacuum state, the effective potential  $V$  must have a stable and non-zero minimum point (vacuum). If expanding the effective potential  $V$  with respect to  $\psi^*\psi$  around the critical temperature  $T_c$  (up to second order), we obtain  $V \simeq -a\psi^*\psi - b(\psi^*\psi)^2/2$ . Thus, the global gauge-invariant action for the pairing field  $\psi$  is given by  $S_G^\psi = \int d^4X_\mu [(i\partial^\mu\psi^*)(-i\partial_\mu\psi) + U_{\text{eff}}]$ , where  $U_{\text{eff}} = -V$  is an effective potential [39].

To discuss the local gauge symmetry of the pairing field  $\psi$ , we replace  $U_G$  by  $U_L$  to make a similar gauge transformation. However, in such case,  $-i\partial_\mu\psi' \neq -iU_L\partial_\mu\psi$ . As a result, we introduce new covariant derivatives of the pairing field  $\psi$ ,  $D_\mu = -i\partial_\mu - \beta B_\mu$ , to realize  $(D_\mu\psi)' = U_L(D_\mu\psi)$  [39], which gives rise to three following equations:

$$-i\partial_\mu(U_L\psi) - \beta B'_\mu U_L\psi = -iU_L\partial_\mu\psi - \beta U_L B_\mu\psi, \text{ c.c.}, \quad (13)$$

and

$$B'_\mu = -\frac{i}{\beta}(\partial_\mu U_L)U_L^\dagger + U_L B_\mu U_L^\dagger, \quad (14)$$

where c.c. is the complex conjugate. With the help of equation (14) and the covariant derivatives  $D_\mu$ , we confirm that  $(D^\mu\psi^*)(D_\mu\psi)$  are invariant under the local gauge transformation  $U_L$ , and so is the effective potential  $U_{\text{eff}}$ . As a consequence, we obtain the space-time action for the pairing field  $\psi$  in the local gauge symmetry,

$$S_L^\psi = \int d^4X_\mu [(D^\mu\psi^*)(D_\mu\psi) + U_{\text{eff}} + \mathcal{L}_V], \quad (15)$$

where  $B_\mu$  are called the  $SU(2)$  Yang-Mills gauge fields,  $\beta$  is a constant reflecting the coupling between the pairing field  $\psi$  and the Yang-Mills gauge fields  $B_\mu$ ,  $\mathcal{L}_V = -V_{\mu\nu}V^{\mu\nu}/4$ , with  $V_{\mu\nu} = D_\mu B_\nu - D_\nu B_\mu$ , is the energy density invariant of the Yang-Mills gauge fields  $B_\mu$ .

Due to the identical gauge properties of the pairing field  $\psi$  and the Fermi atom field  $\phi$ , the Yang-Mills gauge fields  $B_\mu$  must have the same terms as the synthetic non-Abelian gauge fields  $A_\mu$ . Moreover, they have an identical conserved quality called the  $SU(2)$  charge, according to Noether's theorem [39]. This means that  $\alpha = \beta$ . The above two results lead to a significant conclusion that the pairing field  $\psi$  can also couple identically with the non-Abelian gauge fields  $A_\mu$ , imposing originally on the Fermi atoms, and have a similar internal helical doublet, like the Fermi atoms. In addition, we obtain equation (1) in the text, by extracting the spatial part of the space-time action in equation (15).

**The derivation of the first and second  $SU(2)$  GL equations.** The variation of the total free energy functional can be written formally as

$$\delta F_s = \int d^3\mathbf{R}(\delta f_n + \delta U_{\text{eff}} + \delta f_c + \delta f_G). \quad (16)$$

When condensate of the pairing field  $\psi$  occurs,  $\delta f_n \equiv 0$ . Since the effective potential density does not depend on the synthetic non-Abelian gauge fields, we have

$$\delta U_{\text{eff}} = \delta U_{\text{eff}}(\psi) + \delta U_{\text{eff}}(\psi^*). \quad (17)$$

If further neglecting the higher-order terms with respect to  $\delta\psi$  and  $\delta\psi^*$ , we derive  $\delta U_{\text{eff}}(\psi) = a\delta\psi\psi^* + b\delta\psi\psi^*\psi\psi^*$  and  $\delta U_{\text{eff}}(\psi^*) = a\psi\delta\psi^* + b\delta\psi^*\psi^*\psi\psi^*$ .

For the coupled term between the pairing field  $\psi$  and the synthetic non-Abelian gauge fields  $A_i$ , we have

$$\delta f_c = \delta f_c(\psi) + \delta f_c(\psi^*) + \delta f_c(A_i), \quad (18)$$

where  $i$  and  $j$  run over  $x$ ,  $y$ , and  $z$ , because the pairing has a 3D momentum. After a careful calculation, we have  $\delta f_c(\psi) = [(\partial_i^*\psi^* + i\alpha A_i\psi^*)\delta\psi - (\partial_i^* + i\alpha A_i)^2\psi^*\delta\psi]/4m$  and  $\delta f_c(\psi^*) = [(\partial_i\psi - i\alpha A_i\psi)\delta\psi^* - (\partial_i - i\alpha A_i)^2\psi\delta\psi^*]/4m$ . On the other hand, when neglecting the higher-order terms with respect to  $\delta A_i$ , we obtain  $\delta f_c(A_i) \simeq \alpha^2\psi^*\psi\delta A_i A_i/2m + i\alpha(\psi^*\partial_i\psi\delta A_i - \psi\partial_i\psi^*\delta A_i)/4m$ .

Finally, we consider the variation of the energy functional density of the synthetic non-Abelian gauge fields  $A_i$ ,

$$\delta f_G = \varrho [L_{ij}(\mathbf{A} + \delta\mathbf{A})L_{ij}(\mathbf{A} + \delta\mathbf{A}) - L_{ij}(\mathbf{A})L_{ij}(\mathbf{A})], \quad (19)$$

where  $L_{ij}(\mathbf{A}) = -i\partial_i A_j + i\partial_j A_i + \alpha A_j A_i - \alpha A_i A_j$  and  $L_{ij}(\mathbf{A} + \delta\mathbf{A}) = -i\partial_i A_j + i\partial_j A_i - i\partial_i \delta A_j + i\partial_j \delta A_i + \alpha A_j A_i - \alpha A_i A_j - \alpha A_j \delta A_i + \alpha A_i \delta A_j - \alpha \delta A_i A_j + \alpha \delta A_j A_i - \alpha \delta A_i \delta A_j + \alpha \delta A_j \delta A_i$ . When neglecting all the high-order terms, such as  $O^2(\delta A_i)$ ,  $O^2(\partial_i \delta A_i)$ , and  $O^2(\delta A_j \partial_i \delta A_i)$ , we obtain

$$\begin{aligned} \delta f_G &= 2\varrho(\alpha A_i A_j - \alpha A_j A_i + i\partial_i A_j - i\partial_j A_i) \quad (20) \\ &\quad \times (\alpha A_i \delta A_j - \alpha A_j \delta A_i + i\partial_i \delta A_j - i\partial_j \delta A_i \\ &\quad + \alpha \delta A_i A_j - \alpha \delta A_j A_i) \\ &= 2\varrho(\Pi_i A_j - \Pi_j A_i)(\Pi_i \delta A_j - \Pi_j \delta A_i) - \\ &\quad 2\alpha\varrho(\Pi_i A_j - \Pi_j A_i)(\delta A_i A_j - \delta A_j A_i). \end{aligned}$$

Equation (20) shows the properties induced by the synthetic non-Abelian gauge fields  $A_i$ . If all non-commutators vanish, this equation becomes  $\delta f_G = -2\varrho(\partial_i A_j - \partial_j A_i)(\partial_i \delta A_j - \partial_j \delta A_i)$ , which is the typical result for the Abelian gauge field in the  $U(1)$  GL theory.

In the presence of the Abelian gauge fields, we have  $(\nabla \times \mathbf{A}) \cdot (\nabla \times \delta\mathbf{A}) = \delta\mathbf{A} \cdot (\nabla \times \nabla \times \mathbf{A}) - \nabla \cdot [(\nabla \times \mathbf{A}) \times \delta\mathbf{A}]$ . However, in the case of the  $SU(2)$  non-Abelian gauge fields only with the in-plane components (i.e.,  $A_z = 0$ ), the above formula becomes  $\partial_\zeta A_\eta \partial_\zeta \delta A_\eta - \partial_\zeta A_\eta \partial_\eta \delta A_\zeta - \partial_\eta A_\zeta \partial_\zeta \delta A_\eta + \partial_\eta A_\zeta \partial_\eta \delta A_\zeta = (\partial_\eta \partial_\zeta A_\eta - \partial_\zeta^2 A_\zeta - \partial_\eta^2 A_\zeta) \delta A_\zeta + (\partial_\eta^2 A_\zeta \delta A_\zeta + \partial_z^2 A_\zeta \delta A_\zeta + \partial_\zeta A_\eta \partial_\zeta \delta A_\eta - \partial_\zeta A_\eta \partial_\eta \delta A_\zeta + \partial_z A_\eta \partial_z \delta A_\eta - \partial_\zeta \partial_\eta A_\zeta \delta A_\eta)$ , and equation

(20) thus turns into

$$\begin{aligned} \delta f_G = & \varrho[-2(\partial_\eta \partial_\zeta A_\eta - \partial_z^2 A_\zeta - \partial_\eta^2 A_\zeta)\delta A_\zeta + \\ & 4i\alpha[-(\partial_\zeta A_\eta)A_\eta + (\partial_\eta A_\zeta)A_\eta]\delta A_\zeta \\ & -2(\partial_\eta^2 A_\zeta \delta A_\zeta + \partial_z^2 A_\zeta \delta A_\zeta + \partial_\zeta A_\eta \partial_\zeta \delta A_\eta - \\ & \partial_\zeta A_\eta \partial_\eta \delta A_\zeta + \partial_z A_\eta \partial_z \delta A_\eta - \partial_\zeta \partial_\eta A_\zeta \delta A_\eta)]. \end{aligned} \quad (21)$$

In addition, for the 3D momentum of the pairing, the boundary conditions are written as [32]

$$(\partial_i^* + i\alpha A_i)\mathbf{n}\psi^* = 0, \text{ c.c..} \quad (22)$$

Using these boundary conditions, the variation of the total free energy functional is obtained by

$$\delta F_s = \delta F_s(\psi) + \delta F_s(\psi^*) + \delta F_s(A_\zeta), \quad (23)$$

where

$$\begin{aligned} \delta F_s(\psi) = & \int d^3\mathbf{R} \left\{ \frac{1}{4m}[(i\partial_\zeta^* + \alpha A_\zeta)^2 - \partial_z^2]\psi^* \right. \\ & \left. + a\psi^* + b\psi^*\psi^2 \right\} \delta\psi, \end{aligned} \quad (24)$$

$$\begin{aligned} \delta F_s(\psi^*) = & \int d^3\mathbf{R} \left\{ \frac{1}{4m}[(-i\partial_\zeta - \alpha A_\zeta)^2 - \partial_z^2]\psi \right. \\ & \left. + a\psi + b\psi^2\psi \right\} \delta\psi^*, \end{aligned} \quad (25)$$

$$\begin{aligned} \delta F_s(A_\zeta) = & \frac{1}{2m} \int d^3\mathbf{R} (\alpha^2 \psi^* \psi A_\zeta \\ & + \frac{i\alpha}{2} (\psi^* \partial_\zeta \psi - \psi \partial_\zeta \psi^*) \delta A_\mu \\ & + \varrho \int d^3\mathbf{R} \{ -2(\partial_\eta \partial_\zeta A_\eta - \partial_z^2 A_\zeta - \partial_\eta^2 A_\zeta) \delta A_\zeta \\ & + 4i\alpha [-(\partial_\zeta A_\eta)A_\eta + (\partial_\eta A_\zeta)A_\eta] \delta A_\zeta \} \\ & + \varrho \int d^3\mathbf{R} [ -2(\partial_\eta^2 A_\zeta \delta A_\zeta + \partial_z^2 A_\zeta \delta A_\zeta \\ & + \partial_\zeta A_\eta \partial_\zeta \delta A_\eta - \partial_\zeta A_\eta \partial_\eta \delta A_\zeta + \partial_z A_\eta \partial_z \delta A_\eta \\ & - \partial_\zeta \partial_\eta A_\zeta \delta A_\eta) ]. \end{aligned} \quad (26)$$

Finally, using the conditions  $\delta F_s(\psi) = \delta F_s(\psi^*) = 0$ , we obtain the first GL equation (see equation (3) in the text). In addition, by considering  $\delta F_s(A_\zeta) = 0$ , we derive the second GL equation and the supercurrents in the  $x$ ,  $y$ , and  $z$  directions (see equations (5)-(9) in the text).

**The derivation of equation (11).** We rewrite equation (10) as

$$j_\zeta^m = -\frac{a\alpha}{2bd'} \hat{\Xi} \text{Im} \left[ \exp \left( \begin{bmatrix} 0 & \varpi_1 \\ \varpi_2 & 0 \end{bmatrix} \right) \right], \quad (27)$$

where  $\varpi_1 = -i\alpha\lambda_2[Gt + g \sin(\omega_0 t)/\omega_0 + \Delta\varphi_{01}]$  and  $\varpi_2 = -i\alpha\lambda_1[Gt + g \sin(\omega_0 t)/\omega_0 + \Delta\varphi_{02}]$ . When  $\{G, g\} \ll E_F$ ,

we approximately obtain

$$\begin{aligned} j_\zeta^m = & -\frac{a\alpha}{2bd'} \hat{\Xi} \\ & \times \text{Im} \left[ \exp \left( \begin{bmatrix} 0 & \frac{\alpha\lambda_1}{i}(Gt + \Delta\varphi_{01}) \\ \frac{\alpha\lambda_2}{i}(Gt + \Delta\varphi_{02}) & 0 \end{bmatrix} \right) \right] \\ & \times \exp \left( \begin{bmatrix} 0 & \frac{\alpha g \lambda_1}{i\omega_0} \sin(\omega_0 t) \\ \frac{\alpha g \lambda_2}{i\omega_0} \sin(\omega_0 t) & 0 \end{bmatrix} \right) \right]. \end{aligned} \quad (28)$$

Based on the definition of matrix exponential, equation (28) turns into

$$j_\zeta^m = -\frac{a\alpha}{2bd'} \hat{\Xi} \text{Im}(\hat{P}), \quad (29)$$

where  $\hat{P} = \text{diag}(\exp[i\alpha(Gt + \Delta\varphi_{01})] \exp[i\alpha g \sin(\omega_0 t)/\omega_0] \lambda_1 \lambda_2, \exp[i\alpha(Gt + \Delta\varphi_{02})] \exp[i\alpha g \sin(\omega_0 t)/\omega_0] \lambda_1 \lambda_2)$ . To analyze the properties of the gauge invariant mass current, we need take a Fourier-Bessel power series for the elements of the matrix  $\hat{P}$ . By considering the parity of the Bessel function, i.e.,  $J_k(x) = (-1)^k J_{-k}(x)$ , we have

$$\begin{aligned} \exp[i\frac{\alpha g}{\omega_0} \sin(\omega_0 t)] = & \sum_{k=-\infty}^{\infty} J_k\left(\frac{\alpha g}{\omega_0}\right) \cos(k\omega_0 t) + iJ_k\left(\frac{\alpha g}{\omega_0}\right) \sin(k\omega_0 t) \\ = & \sum_{k=-\infty}^{\infty} (-1)^k J_k\left(\frac{\alpha g}{\omega_0}\right) \exp(-ik\omega_0 t). \end{aligned} \quad (30)$$

Substitute equation (30) into the matrix  $\hat{P}$  yields equation (11).

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**Competing Interests** The authors declare that they have no competing financial interests.

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